**Indian Statistical Institute, Bangalore**
B. Math (Hons.) Third Year
Second Semester - Complex Analysis

Midterm Exam — Date: 18th February 2025
Maximum marks: 30 — Duration: 2 hours

**Answer any five, each question carries 6 marks**

1. (a) Prove that a holomorphic function is continuous *(Marks 2)*.

   (b) If $\Omega$ is a region and $f \in H(\Omega)$ with $f' = 0$ on $\Omega$, prove that $f$ is constant.

2. (a) Determine all entire functions whose every power series representation has at least one coefficient zero.

   (b) If $f, g \in H(\mathbb{C})$ with $fg = 0$, prove that $f = 0$ or $g = 0$ *(Marks: 2)*.

3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $R$. Then prove that $f$ is holomorphic and $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ on $|z| < R$.

4. (a) If $P$ is a non-constant polynomial, then each connected component of $\{z \in \mathbb{C} \mid |P(z)| < c\}$ contains a zero of $P$ for any $c > 0$ *(Marks: 4)*.

   (b) Find $\mathrm{Ind}_{\gamma}(z)$ for any closed path $\gamma$ in a convex region $\Omega$ and $z \notin \Omega$.

5. (a) If $u$ is the real part of an analytic function, prove that $u_{xx} + u_{yy} = 0$.

   (b) Determine analytic functions $f$ so that $|f|^2$ is the real part of some analytic function *(Marks: 4)*.

6. (a) Prove that the derivative of an injective analytic function nowhere vanishes.

   (b) Prove that $\int_{\gamma} z^n = 0$ for any closed path $\gamma$ and $n \geq 0$ *(Marks: 2)*.

7. (a) Let $\Omega$ be a bounded region and $f\colon \overline{\Omega} \to \mathbb{C}$ be a continuous function that is analytic on $\Omega$ with $|f(z)| = 1$ on $\partial\Omega$. If $f$ nowhere vanishes, prove that $f$ is constant *(Marks: 3)*.

   (b) Find all analytic functions $f$ for which $\overline{f}$ is also analytic where $\overline{f}(z) = \overline{f(z)}$.